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FUNCTION THEORY OF FINITE ORDER ON ALGEBRAIC VARIETIES. I (B)

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5. The Picard variety with growth conditions

(a) *Line bundles of finite order*

Let A be a smooth affine algebraic variety and Λ one of the λ-rings given by the three examples in § 2 (b). We want to define the *Picard variety* $\mathrm{Pic}_{\Lambda}(A)$ with growth condition.[26] To to this, we will give three definitions of what it means for a holomorphic line bundle $\boldsymbol{L} \to A$ to have order Λ. It is then a basic theorem that the three definitions are equivalent.

Definition A. The holomorphic line bundle $\boldsymbol{L} \to A$ has order Λ if there is a holomorphic mapping

$$f\colon A \to \boldsymbol{P}_N$$

such that (i) f has order Λ, and (ii) $f^{-1}(\boldsymbol{H}) = \boldsymbol{L}$ where $\boldsymbol{H} \to \boldsymbol{P}_N$ is the standard (ample) line bundle.[27]

Definition B. The holomorphic line bundle $\boldsymbol{L} \to A$ has order Λ if there is an analytic divisor $V \subset A$ such that (i) $\boldsymbol{L} = [V]$ is the line bundle determined by V, and (ii) V has order Λ.[28]

The third definition is in terms of transition functions relative to a finite open

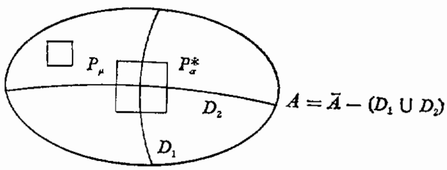

Fig. 3

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[26] The definition and basic properties of $\mathrm{Pic}_{\Lambda}(A)$ can probably be given for a general λ-ring, but we will not try to do this here.

[27] Recall from § 2 (c) that f has order Λ if $f^*[\mathscr{M}(\boldsymbol{P}_N)] \subset \mathscr{M}_{\Lambda}(A)$.

[28] Recall from (2.8) that V has order Λ if the order functions $N(V,r) = O(\lambda(r))$ for some $\lambda \in \Lambda$ and for all punctured polycylinders at infinity.

covering $U = \{P_\mu, P_\alpha^*(k_\alpha)\}$ of A where the P_μ are polycylinders in the interior of A and where the $P_\alpha^*(k_\alpha)$ are punctured polycylinders obtained by intersecting A with a polycylinder P_α on a smooth completion $\bar{A}$ of A.

This definition is somewhat complicated by the fact that, if $k_\alpha \geq 2$, then the restriction $\boldsymbol{L} | P^*(k_\alpha)$ need *not* be analytically trivial (cf. Proposition (4.18)). In order to simplify matters, we shall assume that $A = \bar{A} - D$ is the complement of a smooth ample divisor D on a smooth projective variety $\bar{A}$. With some additional work together with Proposition (4.18), the general case may also be done.[29]

Definition C. The holomorphic line bundle $\boldsymbol{L} \to A$ has order Λ if, relative to a finite open covering $U = \{P_\mu;\ P_\alpha^*\}$ of A by polycylinders P_μ and punctured polycylinders P_α^*, $\boldsymbol{L} \to A$ has transition functions $\{f_{\mu\nu}, f_{\mu\alpha}, f_{\alpha\beta}\}$ such that

$$f_{\alpha\beta} \colon P_\alpha^* \cap P_\beta^* \to \boldsymbol{C}^*$$

has order Λ.[30]

Theorem I. *The above definitions A, B, C are all equivalent.*

(b) *Remarks on the proof of Theorem I*

The proof of Theorem I is, to the author, an instructive iterplay between geometry and function theory. Here is a brief outline of how the arguments run.

Ⓐ ⇒ Ⓑ: Given a holomorphic mapping $f \colon A \to \boldsymbol{P}_N$ which has order Λ, we must find a divisor V on A such that: (i) $[V] = f^{-1}(\boldsymbol{H})$, and (ii) V has order Λ. The natural candidate for V is the inverse image $f^{-1}(\boldsymbol{P}_{N-1})$ of a general hyperplane $\boldsymbol{P}_{N-1} \subset \boldsymbol{P}_N$.

To see that this works, we choose homogeneous coordinates $[\xi_0, \cdots, \xi_N]$ on $\boldsymbol{P}_N$ and set $\phi = f^*(\xi_0/\xi_1)$. Then we may assume that $f^{-1}(\boldsymbol{P}_{N-1})$ is given by $\phi = 0$, where ϕ is a mermorphic function in $\mathscr{M}_\Lambda(A)$. Localizing in a punctured polycylinder P^* at infinity in A, we have by the definition (2.13) and (2.11) of what it means for ϕ to be in $\mathscr{M}_\Lambda(A)$ that $V \cap P^* = \{\eta = 0\}$ where $\eta \in \mathcal{O}_\Lambda(P^*)$ is a holomorphic function of order Λ. The implication Ⓐ ⇒ Ⓑ will then follow from an estimate

[29] From Proposition (4.18), we see that the restriction $\boldsymbol{L} | P^*(k)$ satisfies an equation of holomorphic line bundles

$$\boldsymbol{L} | P^*(k) \cong \prod_{i<j} [l_{ij} D_{ij}] \qquad (l_{ij} \in \boldsymbol{Z}) .$$

By comparing this equation in the overlap of two such punctured polycylinders, we may carry out definition C in general.

[30] Thus we are measuring the growth of $f_{\alpha\beta}$ by its maximum modulus; i.e., its "affinity" for the point $\infty \in \boldsymbol{P}_1$. Equally important, obviously, is the affinity of $f_{\alpha\beta}$ for $0 \in \boldsymbol{P}_1$, i.e., the growth of $f_{\alpha\beta}^{-1}$. It is a consequence of the two facts that (i) $m(f_{\alpha\beta}, r) = m(f_{\alpha\beta}^{-1}, r)$ (Green's theorem), and (ii) $m(f_{\alpha\beta}, r) = 0(M(f_{\alpha\beta}, r)) = 0(m(f_{\alpha\beta}, 2r))$ that the growth of $f_{\alpha\beta}^{-1}$ is determined by that of $f_{\alpha\beta}$.

$$N(V, r) = O(M(\eta, r)) \; ; \tag{5.1}$$

i.e., *we must estimate the area of the zero locus* $\eta = 0$ *in terms of the maximum modulus of the holomorphic function* η. This follows, as in the 1-variable case, from *Jensen's theorem* [19].[31]

Ⓑ ⇒ Ⓒ: Given the divisor $V \subset A$ and the open covering $U = \{P_\mu; P_\alpha^*\}$ of A, we may write the intersection

$$V \cap P_\alpha^* = \{\eta_\alpha = 0\} \tag{5.2}$$

as the zeroes of some holomorphic function $\eta_\alpha \in \mathcal{O}(P_\alpha^*)$. The transition functions of the line bundle $L \to A$ are then the ratios

$$f_{\alpha\beta} = \eta_\alpha / \eta_\beta \; , \tag{5.3}$$

together with the $f_{\mu\nu}$ and $f_{\mu\alpha}$ which do not concern us. From (5.3) it follows that, in order to prove the implication Ⓑ ⇒ Ⓒ, we must show: *if V has order Λ, then the holomorphic function η_α in* (5.3) *may also be chosen to have order* Λ. In 1-variable, this is a well-known theorem of Hadamard, which is proved using properties of this *Weierstrass primary factors* $E(\mu, q)$ (cf. [19, pp. 221–229]), and in several variables, if P^* is given by

$$\{(z, w) \in \boldsymbol{C} \times \boldsymbol{C}^{n-1} \colon 0 < |z| \leq 1, |w_j| \leq 1\} \; ,$$

then we may use the 1-variable result in the punctured discs Δ_w^* given by $w =$ constant (cf. [24] for a similar method).[32] Alternatively, with a little work we may adapt the potential-theoretic methods of Lelong [18] to the case at hand.

Ⓒ ⇒ Ⓑ: Suppose now that $L \to A$ has transition functions as in definition C, and let $i \colon A \to \bar{A}$ be the inclusion. Then we may define a sub-sheaf

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[31] We may formalize Jensen's theorem as follows: Let $\mathrm{Div}_\Lambda^+(A)$ be the group of *effective* divisors on A which have order Λ. Then by Jensen's theorem there is a homomorphism

$$\delta \colon \mathcal{M}_\Lambda(A) \to \mathrm{Div}_\Lambda^+(A) \tag{5.2}$$

given by sending $\phi \in \mathcal{M}_\Lambda(A)$ into the hypersurface $\phi = 0$ on A. It will be a consequence of the results in § 5 (c) below that we have an exact sequence (cf. Proposition (5.26))

$$\mathcal{O}_\Lambda(A) \xrightarrow{\delta} \mathrm{Div}_\Lambda^+(A) \longrightarrow H^2(A, \boldsymbol{Z}) \longrightarrow 0 \; ,$$

where $\mathrm{Div}_\Lambda^+(A) \to H^2(A, \boldsymbol{Z})$ is the homology class mapping, provided that Λ contains the ring of all polynomials in r (cf. Theorem II below).

[32] In the case $n = 2$, what we do is the following: By choosing P^* correctly we may assume that all intersections

$$V \cdot \Delta_w = \{z_1(w), z_2(w), \cdots\}$$

are divisors on the punctured Z-disc such that

$$\sum_{j=1}^{\infty} \int_{|w| \leq 1} |z_j(w)|^q \, dw d\bar{w} < \infty$$

for some integer $q > 0$, and where all $|z_j(w)| \leq 1 - \varepsilon$ for some $\varepsilon > 0$. Then $\eta(z, w) = \prod_{j=1}^{\infty} E(z_j(w)/z, q)$ gives the required function.

$$\mathcal{O}_\Lambda(L) \subset i_* \mathcal{O}_{\text{hol}}(L)$$

of the direct image $i_*\mathcal{O}_{\text{hol}}(L)$ whose sections over an open set $U \subset \bar{A}$ are the holomorphic sections of order Λ over $U^* = U \cap A$. (For example, if Λ is given by example 1 in § 2 (b), then it is easy to see that we may choose $\bar{A}$ such that there is a holomorphic line bundle $\bar{L} \to \bar{A}$ with $\bar{L}|A = L$, and $\mathcal{O}_\Lambda(L)$ is just the sections of $\bar{L}$ with finite poles along $\bar{A} - A$.)

To construct a holomorphic mapping $f: A \to P_N$ which has order Λ and which realizes $L \to A$ as $f^{-1}(H)$, essentially we must find enough global sections in

$$H^0(\bar{A}, \mathcal{O}_\Lambda(L)) .$$

This in turn results from the following basic

(5.4) Theorem B with Λ-growth conditions. *For any algebraic vector bundle $E \to A$ we have the vanishing theorem*

$$H^q(\bar{A}, \mathcal{O}_\Lambda(L \otimes E)) = 0 \qquad (q > 0) .$$

Outline of the proof. To define the sheaf $\mathcal{O}_\Lambda(L \otimes E)$, we first let $\mathcal{O}_\Lambda(E) = \mathcal{O}_\Lambda \otimes_{\mathcal{O}_{\text{alg}}} \mathcal{O}_{\text{alg}}(E)$ where $\mathcal{O}_{\text{alg}}(E) \subset i_*\mathcal{O}_{\text{hol}}(E)$ are the sections of E which have finite poles along $\bar{A} - A$. Then we may let $\mathcal{O}_\Lambda(L \otimes E) = \mathcal{O}_\Lambda(L) \otimes_{\mathcal{O}_\Lambda} \mathcal{O}_\Lambda(E)$.

We observe that Theorem (5.4) gives

$$H^q(\bar{A}, \mathcal{O}_\Lambda(L) \otimes_{\mathcal{O}_{\text{alg}}} S) = 0 \qquad (q > 0)$$

for any coherent algebraic sheaf $S \to \bar{A}$. Taking S to be $\mathcal{O}_{\text{alg}}/\boldsymbol{m}_x$ ($\boldsymbol{m}_x \subset \mathcal{O}_x$ being the maximal ideal of $x \in A$), we obtain global sections σ of $\mathcal{O}_\Lambda(L)$ with $\sigma(x) \neq 0$.

The proof of our theorem proceeds in several steps.

Step 1. We first choose a family $\tau_\Lambda = \{\tau_{\Lambda,q}\}$ ($q \in Z^+$) of strongly-pseudo-convex exhaustion functions which are suitable for the problem. Referring to examples (i), (ii) of λ-rings in § 2(b) (the algebraic and finite order cases), we may let respectively

(5.5)
$$\tau_{\Lambda,q} = q \log (1 + |z_1|^2 + \cdots + |z_N|^2) ,$$
$$\tau_{\Lambda,q} = |z_1|^{2q} + \cdots + |z_N|^{2q} ,$$

where $z_1, \cdots, z_N$ are the Euclidean coordinates relative to an embedding $A \subset C^N$ as a smooth affine variety. Localizing in a punctured polycylinder

$$P^* = \{(z, w) \in C \times C^{m-1}: 0 < |z| \leq 1, |w| \leq 1\}$$

at infinity in A, we have that approximately

(5.5)*
$$\tau_{\Lambda,q} \sim -\log |z|^{2q} , \qquad \tau_{\Lambda,q} \sim |z|^{2q}$$

in the sense that each side is "0" of the other. It follows from (5.5)* that the holomorphic functions in P^* which have order Λ are exactly those $\eta \in \mathcal{O}(P^*)$ such that

$$\int_{P^*} |\eta|^2 e^{-\tau_{\Lambda,q}} d\mu < \infty \qquad (5.6)^*$$

for some $q > 0$, and where $d\mu$ is Euclidean measure on $P^* \subset \boldsymbol{C}^n$. If we let Φ denote the restriction to P^* of a volume form on $\bar{A}$, then $\Phi \sim d\mu$ and we shall rewrite (5.6)* as

$$\int_{P^*} |\eta|^2 e^{-\tau_\Lambda} \Phi < \infty \qquad (5.6)$$

(this notation means that for large q we have (5.6)*).

Step 2. We now discuss the groups $H^q(\bar{A}, \mathcal{O}_\Lambda(A))$, where $\boldsymbol{L}$ and $\boldsymbol{E}$ are assumed to be trivial. Choose a Kähler metric $ds^2_{\bar{A}} = \sum_{i,j} g_{i\bar{j}} d\mu_i d\bar{\mu}_j$ on $\bar{A}$, and let Φ be the corresponding volume form. Denote by $\mathscr{C}^{0,q}$ the sheaf of $\boldsymbol{C}^\infty$ $(0, q)$ forms on A, and define the subsheaf

$$\mathscr{C}^{0,q}_\Lambda \subset i_*(\mathscr{C}^{0,q})$$

by letting the sections of $\mathscr{C}^{0,q}_\Lambda$ over a punctured polycylinder P^* be the C^∞ $(0, q)$ forms ξ on P^* which satisfy the L^2-conditions

$$\int_{P^*} \|\xi\|^2 e^{-\tau_\Lambda} \Phi < \infty \ , \qquad \int_{P^*} \|\bar{\partial}\xi\|^2 e^{-\tau_\Lambda} \Phi < \infty \ .$$

This gives us a complex of sheaves

$$0 \longrightarrow \mathcal{O}_\Lambda(A) \longrightarrow \mathscr{C}^{0,0}_\Lambda \xrightarrow{\bar{\partial}} \mathscr{C}^{0,1}_\Lambda \xrightarrow{\bar{\partial}} \cdots .$$

We then use the methods of Hörmander [15, Chapter IV], to show that the cohomology sheaves

$$i^q_*(\mathscr{C}^{0,*}_\Lambda) = \{\ker \bar{\partial} : \mathscr{C}^{0,q}_\Lambda \to \mathscr{C}^{0,q+1}_\Lambda\} / \bar{\partial}\mathscr{C}^{0,q-1}_\Lambda$$

are zero for $q > 0$. (This is the Poincaré lemma for the above complex of sheaves.) If we let

$$\mathscr{C}^{0,q}_\Lambda = H^0(\bar{A}, \mathscr{C}^{0,q}_\Lambda) \ ,$$
$$H^{0,q}_\Lambda(\bar{A}) = \{\ker \bar{\partial} : \mathscr{C}^{0,q}_\Lambda \to \mathscr{C}^{0,q+1}_\Lambda\} / \bar{\partial}\mathscr{C}^{0,q-1}_\Lambda \ ,$$

then it follows that

$$H^q(\bar{A}, \mathcal{O}_A(A)) \cong H_A^{0,q}(\bar{A}) \ .$$

Thus far we have only used the exhaustion function τ_A locally at infinity, and have made no use of the fact that it exists globally on A. This is now utilized when we appeal again to the methods of [15, Chapter IV] to show that[33]

$$H_A^{0,q}(\bar{A}) = 0 \qquad (q > 0) \ .$$

Step 3. To discuss the groups $H^q(\bar{A}, \mathcal{O}_A(L \otimes E))$, we shall assume for simplicity that E is trivial. In a sufficiently small punctured polycylinder P^* at infinity, we have $L \mid P^* \cong P^* \times C$. Therefore the above discussion carries over completely as far as the local aspects are concerned. The problem is that, for the global case, we must choose an Hermitian metric along the fibres of $L \rightarrow A$. Relative to a trivialization $L \mid P^* \cong P^* \times C$, the holomorphic sections of L will be holomorphic functions η, but the norm as a section of L will now be

$$\|\eta(z)\|^2 = a(z) \, |\eta(z)|^2 \ ,$$

where $a(z) > 0$ is the metric. A first condition on our metric is that we should have

$$(\alpha) \qquad a(z) e^{-\tau_A(z)} \sim e^{-\tau_A(z)} \ ,$$

so that the sections in $\mathcal{O}_A(L)$ are still characterized by essentially the same L^2-growth conditions as before. The second condition on our metric is somewhat more subtle. Namely, an essential ingredient in the Hörmander estimates is played by the *E. E. Levi form* $dd^c\tau_A$. The effect of the metric in L is to modify $dd^c\tau_A$ additively by the *curvature form* $\theta_L = dd^c \log a(z)$. Thus a second condition on our metric is

$$(\beta) \qquad dd^c\tau_A \pm \theta_L \sim dd^c\tau_A \ .$$

To conclude our outline of the proof of Theorem (5.4), we shall therefore prove the

Lemma. *In the finite order case* ($A = \{1, r\}$), *we may choose a metric in* $L \rightarrow A$ *so that* (α) *and* (β) *above are satisfied.*

Proof. Let $\{P_\alpha^*, P_\mu\}$ be a covering of A by punctured polycylinders, and $\{f_{\alpha\beta}, f_{\alpha\mu}, f_{\mu\nu}\}$ the (finite order) transition functions of L relative to this covering. If $h_\alpha = 0$ is a local defining equation for $(\bar{A} - A) \cap P_\alpha$, then

$$f_{\alpha\beta} = e^{(g/h_\alpha^k)} \cdot (h_\alpha)^l \cdot \mu \ ,$$

where g and μ are holomorphic in P_α. Thus $\log |f_{\alpha\beta}| |h_\alpha|^m$ is of class $C^{(2)}$ in P

[33] The author is indebted to P. Deligne for showing the author some notes of his on the realization of algebraic sheaf cohomology as $\bar{\partial}$-cohomology with growth conditions.

for m sufficiently large. Choose a global C^∞ function $\rho \geq 0$ on $\bar{A}$ such that $\rho \sim |h_\alpha|^m$ in P_α. Then $\theta_{\alpha\beta} = \rho \cdot \log |f_{\alpha\beta}|$ is of class $C^{(2)}$ and

$$\theta_{\alpha\beta} + \theta_{\beta\gamma} = \theta_{\alpha\gamma}$$

in $P_\alpha \cap P_\beta \cap P_\gamma$. By a partition of unity, we may find $C^{(2)}$ functions b_α in P_α such that

$$\rho \log |f_{\alpha\beta}| = b_\alpha - b_\beta$$

in $P_\alpha \cap P_\beta$. Then we may define our metric by

$$a_\alpha(z) = e^{(b_\alpha/\rho)} .$$

Since $\tau_{\Lambda,q} | P_\alpha \sim 1/|h_\alpha|^q$, it follows that (α) is satisfied. Futhermore, the curvature

$$\theta_L = dd^c \log a_\alpha = c_\alpha/\rho^2 ,$$

where c_α is bounded. From this it follows that (β) is satisfied, provided that we add to $\tau_{\Lambda,q}$ in (5.5) a term $|P_1(z)|^2 + \cdots + |P_k(z)|^2$ for suitable polynomials in $P_\alpha(z)$.

Remark. Let $\Gamma_\Lambda(L) = H^0(\bar{A}, \mathcal{O}_\Lambda(L))$ be the global holomorphic sections of $L \to A$ with Λ-growth conditions, and set $\Gamma(L) = H^0(A, \mathcal{O}_{\text{hol}}(L))$. Then, with the topology of uniform convergence on compact sets $K \subset A$, $\Gamma_\Lambda(L)$ is a complete topological vector space. We shall write

$$\Gamma_\Lambda(L) = \bigcup_{q=1}^{\infty} \Gamma_{\Lambda,q}(L)$$

as an increasing union of Hilbert spaces $\Gamma_{\Lambda,q}(L)$ such that each mapping

$$\Gamma_{\Lambda,q}(L) \to \Gamma(L)$$

is completely continuous (bounded sets go into compact sets). For this we let

$$\Gamma_{\Lambda,q}(L) = \left\{\eta \in \Gamma(L) : \int_A \|\eta\|^2 e^{-\tau_{\Lambda,q}} \Phi < \infty\right\} .$$

Letting Ψ_q be the measure $e^{-\tau_{\Lambda,q}}\Phi$ on A, we let

$$(\eta, \xi) = \int_A (\eta, \xi)_z \Psi_q(z) \qquad (\eta, \xi \in \Gamma_{\Lambda,q}(L)) ,$$

and this converts $\Gamma_{\Lambda,q}(L)$ into a complete Hilbert space with the required properties.

Example. In case $A = C^n$, L is the trivial bundle, and $\Lambda = R^+$ are the algebraic growth conditions, $\Gamma_{\Lambda,q}(L)$ is just the finite dimensional vector space of polynomials of degree $q - n$.

(c) *Definition of* $\mathrm{Pic}_\Lambda(A)$ *and Theorem II*

Given a line bundle $L \to A$ which is of order Λ, it follows from footnote 30 that the dual L^* is also of order Λ. Therefore, if both L and L' are of order Λ, we may define the sheaf

$$\mathcal{O}_\Lambda(\mathrm{Hom}(L, L')) = \mathcal{O}_\Lambda(L^* \otimes L') ,$$

and use the global sections of this sheaf to say when L and L' are Λ-isomorphic.

We will now give two equivalent definitions of $\mathrm{Pic}_\Lambda(A)$, the *Picard variety with Λ-growth conditions.*

Definition α. $\mathrm{Pic}_\Lambda(A)$ is given by the Λ-isomorphism classes of holomorphic line bundles of order Λ on A.

Definition β. $\mathrm{Pic}_\Lambda(A)$ is the group $\mathrm{Div}_\Lambda(A)$ of divisors of order Λ on A modulo principle divisors (ϕ) where $\phi \in \mathcal{M}_\Lambda(A)$.[34]

(5.7) Proposition. *The above definitions of* $\mathrm{Pic}_\Lambda(A)$ *are equivalent.*

We want to see how $\mathrm{Pic}(A)$ varies with Λ. Referring to the three examples in § 2(b) of λ-rings, $\mathrm{Pic}_\Lambda(A)$ will be denoted respectively by

$$\begin{aligned} &\mathrm{Pic}_{\mathrm{alg}}(A) && (\Lambda = R^+) , \\ &\mathrm{Pic}_{\mathrm{f.o.}}(A) && (\Lambda = \{1, r\}) , \\ &\mathrm{Pic}_{\mathrm{hol}}(A) && (\Lambda = \text{all functions}) . \end{aligned}$$

If $\Lambda_1 \subset \Lambda_2$, then there is an obvious map

$$\mathrm{Pic}_{\Lambda_1}(A) \to \mathrm{Pic}_{\Lambda_2}(A) .$$

Referring to the examples in § 4, we see that the mapping

$$\mathrm{Pic}_{\mathrm{alg}}(A) \to \mathrm{Pic}_{\mathrm{hol}}(A) \tag{5.8}$$

is, in general, neither injective nor surjective. On the other hand, denoting $H^2(A, Z)$ by $\mathrm{Pic}_{\mathrm{top}}(A)$, it follows from § 3 that

$$\mathrm{Pic}_{\mathrm{hol}}(A) \to \mathrm{Pic}_{\mathrm{top}}(A) \tag{5.9}$$

is an isomorphism.

Theorem II. (i) *The group* $\mathrm{Pic}_\Lambda(A)$ *is functorially associated to the algebraic structure on A.* (ii) *If Λ is any λ-ring which contains $\{1, \lambda\}$, then the mapping*

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[34] We recall that the group $\mathrm{Div}_\Lambda^+(A)$ of *effective divisors of order Λ* was defined in § 2(b), and $\mathrm{Div}_\Lambda(A)$ is the group of divisors $V = V_1 - V_2$ where $V_1, V_2 \in \mathrm{Div}_\Lambda^+(A)$ (cf. footnote 31).

$$\mathrm{Pic}_A(A) \to \mathrm{Pic}_{\mathrm{top}}(A)$$

is an isomorphism. (iii) *Finally,* $\{1, r\}$ *is the smallest* λ*-ring with this property.*

Remark. This theorem establishes the *Oka principle for function theory of finite order for the relatively simple case of divisors.* Moreover, it follows from this result that function theory of finite order is the smallest category for which the Oka principle might possibly hold in the general case (use Künneth on the case of divisors).

The Proof of Theorem II is deceptively simple, once we have established Theorem I (§ 5(a)) which essentially says that everything makes sense. We will outline the argument that the mapping

$$\mathrm{Pic}_{\mathrm{f.o.}}(A) \to \mathrm{Pic}_{\mathrm{hol}}(A)$$

is an isomorphism in the case $A = \bar{A} - D$ where D is smooth and ample. First we make three comments:

(i) Let $\mathcal{O}(\mathrm{f.o.})$ be the sheaf on $\bar{A}$ of meromorphic functions with poles of finite order along D, and $\mathcal{O}(\infty) = i_*(\mathcal{O}_A)$ be the sheaf of meromorphic functions with arbitrary singularities along D. Then (cf. [2])

$$(5.10) \qquad \begin{aligned} H^q(\bar{A}, \mathcal{O}(\mathrm{f.o.})) &= 0 \qquad (q > 0)\ , \\ H^q(\bar{A}, \mathcal{O}(\infty)) &= 0 \qquad (q > 0)\ . \end{aligned}$$

(ii) Let $S \to A$ be a sheaf of abelian groups, and $i^q_*(S)$ the q^{th}-direct image of S for the inclusion mapping $i\colon A \to \bar{A}$. Then there is a spectral sequence which abuts to $H^*(A, S)$ and whose E_2-term is

$$E_2^{p,q} = H^p(\bar{A}, i^q_*(S))\ .$$

We shall use this in the case $S = \mathcal{O}^*_A$. Since $i^q_*(\mathcal{O}^*_A) = 0$ for $q > 0$, the spectral sequence is trivial and gives

$$H^1(\bar{A}, i_*(\mathcal{O}^*_A)) \cong H^1(A, \mathcal{O}^*_A)\ .$$

(This spectral sequence is not trivial in the general case when $\bar{A} - A$ may not be smooth. Utilizing it we may relate the local Pic_A's at infinity to the global Pic_A's, and using Proposition (4.18), may then prove Theorem II in the general case.)

(iii) Suppose that $f = 0$ is a local defining equation for D in an open set $U \subset \bar{A}$, and let $\mathcal{O}^*_{\mathrm{f.o.}}$ be the sub-sheaf of $i_*(\mathcal{O}^*_A)$ generated over U by functions of the form

$$\phi = e^{(h/f^k)} \cdot f^l \cdot u\ ,$$

where h and u are holomorphic in all of U with $u \neq 0$. Then, from Theorem I we have

$$\text{(5.11)} \qquad \operatorname{Pic}_{\text{f.o.}}(A) \cong H^1(\bar{A}, \mathcal{O}^*_{\text{f.o.}}) \ .$$

To outline the proof of theorem II, we consider the exact sheaf sequences:

$$\text{(5.12)} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}_{\bar{A}} & \longrightarrow & \mathcal{O}(\text{f.o.}) & \longrightarrow & \exp[\mathcal{O}(\text{f.o.})] & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Z}_{\bar{A}} & \longrightarrow & \mathcal{O}(\infty) & \longrightarrow & \exp[\mathcal{O}(\infty)] & \longrightarrow & 1 \end{array}$$

Using (5.10) we obtain from the cohomology sequences of (5.12) the isomorphisms

$$\text{(5.13)} \qquad H^1(\bar{A}, \exp[\mathcal{O}(\text{f.o.})]) \cong H^1(\bar{A}, \exp[\mathcal{O}(\infty)]) \cong H^2(\bar{A}, \mathbf{Z}) \ .$$

On the other hand, there is a mapping

$$i_*(\mathcal{O}^*_A) \xrightarrow{\text{Res}} \mathbf{Z}_D$$

given by sending ϕ into the Poincaré residue of $d \log \phi$ along D. This gives

$$\text{(5.14)} \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & \exp[\mathcal{O}(\text{f.o.})] & \longrightarrow & \mathcal{O}^*_{\text{f.o.}} & \xrightarrow{\text{Res}} & \mathbf{Z}_D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \exp[\mathcal{O}(\infty)] & \longrightarrow & i_*(\mathcal{O}^*_A) & \xrightarrow{\text{Res}} & \mathbf{Z}_D & \longrightarrow & 0 \ . \end{array}$$

Combining the cohomology sequences of (5.14) together with (5.13) and (5.11) gives

$$\text{(5.15)} \qquad \cdots \to H^2(\bar{A}, \mathbf{Z}) \to \operatorname{Pic}_{\text{f.o.}}(A) \to H^1(D, \mathbf{Z}) \to \cdots \ .$$

By comparing (5.15) with the standard exact sequence

$$\cdots \to H^2(\bar{A}, \mathbf{Z}) \to H^2(A, \mathbf{Z}) \to H^1(D, \mathbf{Z}) \to \cdots ,$$

we arrive at the proof of (ii) in Theorem II.

(d) $\operatorname{Pic}_{\text{f.o.}}(A)$ *and Hodge structure*

In algebraic geometry the basic final objects of study are, without doubt, the smooth projective varieties. One may study these by either algebraic or analytic methods and arrive at the same conclusions (G.A.G.A.). The main advantage in using the underlying complex analytic and C^∞ structures is that these are much closer to the topological properties and, most importantly, lead to the *Hodge structure* on cohomology [25]. For affine varieties, however, the holomorphic functions contain much more information than do the rational functions, and the main goal of this study has been to try in general to interpolate between the analytic and algebraic categories. One of our basic premises is that the function theory of finite order should satisfy the Oka principle and should, therefore, contain all of the topological information. Our second basic premise is that there should be step-by-step obstruction theory going from function theory of finite order to algebraic function theory, and such that the successive obstructions are measured by the Hodge structure on cohomology. The first of these premises has been illustrated in the preceding section, and we shall now give a special case of the second.

Thus, let $\bar{A}$ be a smooth, projective variety and D a sufficiently ample (to be made precise later) smooth divisor on $\bar{A}$. Then the complement $A = \bar{A} - D$ is a smooth affine variety and, from Theorem II, we have

$$\text{(5.16)} \qquad \mathrm{Pic}_{\mathrm{f.o.}}(A) \cong \mathrm{Pic}_{\mathrm{top}}(A) \ .$$

Given a line bundle $\boldsymbol{L} \in \mathrm{Pic}_{\mathrm{f.o.}}(A)$, there exist divisors V of finite order such that $[V] = \boldsymbol{L}$. For each such divisor V we may define *the order* $\rho(V)$ by

$$\text{(5.17)} \qquad \rho(V) = \overline{\lim_{r\to\infty}} (\log N(V, r)/\log r) \ .$$

Then $0 \leq \rho(V) < +\infty$ and $\rho(V) = 0$ if V is algebraic, but *not* conversely (cf. Proposition (4.20)). We then define the order

$$\text{(5.18)} \qquad \rho(\boldsymbol{L}) = \inf_{[V]=\boldsymbol{L}} \rho(V) \ .$$

(5.19) Proposition. *Let V be a divisor of finite order and $\xi = d\log V \in H^1(A, \Omega_c^1) \cong H^2(A, \boldsymbol{C})$ the corresponding cohomology class. If the residue* Res $\xi \in H^1(D, \boldsymbol{C})$ *is zero, then we may find a divisor V' such that*: (i) $\rho(V') \leq \rho(V)$ *and* $[V'] = [V]$ *in* $\mathrm{Pic}_{\mathrm{f.o.}}(A)$; *and* (ii) $\rho(V') = 0 \Leftrightarrow V'$ *is algebraic.*

Outline of proof. Let $\{P_\mu, P_\alpha^*\}$ be a covering of A by polycylinders and punctured polycylinders. Then we will have

$$V \cap P_\alpha^* = \{\eta_\alpha = 0\} \ ,$$

where η_α is a holomorphic function in P_α^* of finite order $\rho(V)$. The transition functions of $[V]$ are $f_{\alpha\beta} = \eta_\alpha/\eta_\beta$ (plus the $f_{\mu\nu}$ and $f_{\mu\alpha}$). If $f_\alpha = 0$ is a local equation for $V \cap P_\alpha$, then we will have

$$f_{\alpha\beta} = e^{(h_{\alpha\beta}/(f_\alpha)^k)} \cdot \mu_{\alpha\beta} \cdot (f_\alpha)^{l_{\alpha\beta}} \ ,$$

where $k = [\rho(V)]$ is the largest integer in $\rho(V)$. It follows that

$$l_{\alpha\beta} + l_{\beta\gamma} = l_{\alpha\gamma}$$

in $P_\alpha \cap P_\beta \cap P_\gamma$, and the cocycle $\{l_{\alpha\beta}\} \in H^1(D, \boldsymbol{Z})$ is the residue of $d\log V$ in $H^1(D, \boldsymbol{C})$. If this class is zero, then we may find new transition functions $\{f'_{\mu\nu}, f'_{\mu\alpha}, f'_{\alpha\beta}\}$ for $[V]$ of the form

$$f'_{\alpha\beta} = e^{(h_{\alpha\beta}/(f_\alpha)^k)} \cdot \mu'_{\alpha\beta} .$$

Then we may find a divisor V' on A with $\rho(V') = k$ and $[V'] = [V]$ (cf. Theorem (5.4)). Furthermore, if $k = 0$, then we may take V' to be algebraic. q.e.d.

This proposition suggests that we let $\overline{\mathrm{Pic}}_{\mathrm{f.o.}}(A) \subset \mathrm{Pic}_{\mathrm{f.o.}}(A)$ be the image of

$$\mathrm{Pic}_{\mathrm{top}}(\bar{A}) \to \mathrm{Pic}_{\mathrm{top}}(A)$$

under the isomorphism (5.16). We then set

$$\overline{\mathrm{Pic}}_{\mathrm{f.o.}}(A)_k = \{L \in \overline{\mathrm{Pic}}_{\mathrm{f.o.}}(A) : \rho(L) \leq k\} . \tag{5.20}$$

Theorem III. (i) *The groups* $\overline{\mathrm{Pic}}_{\mathrm{f.o.}}(A)_k$ *give a filtration of* $\overline{\mathrm{Pic}}_{\mathrm{f.o.}}(A)$ *by non-negative integers* k;[35] (ii) *a line bundle* $L \in \overline{\mathrm{Pic}}_{\mathrm{f.o.}}(A)_0 \Leftrightarrow L$ *is algebraic*; *and* (iii) *the successive quotients have injective maps*

$$d\log : \overline{\mathrm{Pic}}_{\mathrm{f.o.}}(A)_k / \overline{\mathrm{Pic}}_{\mathrm{f.o.}}(A)_{k-1} \to \sum_{\substack{p-q \geq k \\ p+q=2}} H^{p,q}(\bar{A}) . \tag{5.21}$$

Remarks. Statements (i) and (ii) follow from the proof of Proposition (5.19). Implicit in (5.21) is the assertion that the filtration $\{\overline{\mathrm{Pic}}_{\mathrm{f.o.}}(A)_k\}$ is the 2-step filtration

$$\overline{\mathrm{Pic}}_{\mathrm{f.o.}}(A)_0 \subset \overline{\mathrm{Pic}}_{\mathrm{f.o.}}(A)_1 ,$$

and the mapping (5.21) is

$$d\log : \overline{\mathrm{Pic}}_{\mathrm{f.o.}}(A)_1 / \overline{\mathrm{Pic}}_{\mathrm{f.o.}}(A)_0 \to H^{2,0}(\bar{A}) . \tag{5.21$'$}$$

The proof of (iii) is by a careful modification of the proof of Proposition (5.19). The assumption that D is sufficiently ample appears in the mechanism of the proof by use of the vanishing theorem

$$H^1(\bar{A}, \mathcal{O}_{\bar{A}}[lD]) = 0 \qquad (l > 0) .$$

(e) *Some special cases*

We want to give some special cases of Theorems I–III above. These results will contain proofs of the examples discussed in § 4 above.

(i) *Results on curves.* Let A be an affine algebraic curve represented as a branched covering $\pi : A \to \mathbf{C}$ over the z-plane. Given a positive divisor $\delta = \{x_1, x_2, \cdots\}$ on A, we define the *exponent of convergence* $\sigma(\delta)$ to be

$$\sigma(\delta) = \left\{ \inf\{\lambda\} : \sum_{n=1}^{\infty} |\pi(x_n)|^{-\lambda} < \infty \right\} .$$

[35] Cf. statement (ii) in Proposition (4.1).

By a variant of Jensen's theorem, if $f \in \mathcal{O}(A)$ is holomorphic on A, then

$$\sigma(f) \leq \rho(f) \; ; \tag{5.22}$$

i.e., the exponent of convergence of the divisor $f = 0$ is bounded by the order of f.

(5.23) Proposition. *We may always find $f \in \mathcal{O}(A)$ with $(f) = \delta$ and $\rho(f) \leq \max\{\sigma(\delta), 2g\}$ where g is the genus of A. Moreover, this is the best possible estimate in general.*

(5.24) Corollary (*Hadamard*). *If A is a rational curve* ($g = 0$) *then we may find $f \in \mathcal{O}(A)$ with $(f) = \delta$ and $\rho(f) = \sigma(\delta)$.*

Our next result deals with the field $\mathcal{M}_\rho(A)$ of meromorphic functions of finite order ρ on A. Let $\mathcal{O}_\rho(A)$ denote the ring of holomorphic functions of order ρ on A.

(5.25) Proposition. *If $\rho \geq 2g$, then $\mathcal{M}_\rho(A)$ is the quotient field of $\mathcal{O}_\rho(A)$, and this is the best possible general estimate.*

Corollary. *If A is a rational curve, then $\mathcal{M}_\rho(A)$ is the quotient field of $\mathcal{O}_\rho(A)$ for any ρ* (cf. [21]).

(ii) *Function fields on general affine varieties.* Let A be a smooth affine algebraic variety, $\mathcal{M}_{\text{f.o.}}(A)$ the field of meromorphic functions of finite order on A, and $\mathcal{O}_{\text{f.o.}}(A)$ the ring of holomorphic functions in $\mathcal{M}_{\text{f.o.}}(A)$. Finally, denote by $\mathcal{O}_{\text{f.o.}}(A)/\mathcal{O}_{\text{f.o.}}(A)$ the quotient field of $\mathcal{O}_{\text{f.o.}}(A)$.

(5.26) Proposition. (i) *We have an exact sequence*

$$0 \to \mathcal{O}_{\text{f.o.}}(A)/\mathcal{O}_{\text{f.o.}}(A) \to \mathcal{M}_{\text{f.o.}}(A) \to H^2(A, Z) \to 0 \, .$$

(ii) *There exist functions $f_1, \cdots, f_b \in \mathcal{M}_{\text{f.o.}}(A)$* ($b = \operatorname{rank}(H^2(A, Z))$) *such that every $f \in \mathcal{M}_{\text{f.o.}}(A)$ satisfies an equation*

$$f^k = (f_1)^{k_1} \cdots (f_b)^{k_b}(g/h) \qquad (g, h \in \mathcal{O}_{\text{f.o.}}(A)) \, . \tag{5.27}$$

Furthermore, we may take $k = 1$ in (5.27) $\Leftrightarrow$ *$H^2(A, Z)$ has no torsion.*

Remark. The functions of finite order is the smallest class of functions with A-growth conditions for which Proposition (5.26) is true in general.

(5.28) Corollary (*cf.* [21] *and* [24]).

$$\mathcal{M}_{\text{f.o.}}(C^n) = \mathcal{O}_{\text{f.o.}}(C^n)/\mathcal{O}_{\text{f.o.}}(C^n) \, .$$

(iii) *The Lefschetz theorem again.* It is clear that Theorem III contains the Lefschetz theorem [17] to the effect that a cohomology class $\xi \in H^2(\bar{A}, Z)$ is algebraic $\Leftrightarrow \xi^{(2,0)} = 0$ in $H^{2,0}(\bar{A})$. Our proof, which is very close in fact to the original Lefschetz proof, consists of making ξ analytic on a affine open set $A \subset \bar{A}$, and then showing that the Hodge condition $\xi^{(2,0)} = 0$ precisely allows one to take the closure in $\bar{A}$ of the analytic divisor on the affine set A and to obtain an algebraic subvariety of $\bar{A}$.

6. *K*-theory of finite order

This section is mostly conjectural. The author's purpose is to isolate the questions in complex function theory whose resolution would be a major step towards allowing us to do "K-theory of finite order" in a good fashion similar to the finite order Picard variety discussed in the section above. Thus, given a λ-ring Λ, we should like to assign to each smooth, quasi-projective variety A a ring[36] $K_\Lambda(A)$ which has the following properties:

(i) $K_\Lambda(A)$ is functorial for the *algebraic* structure[37] on A.

(ii) For the three λ-rings in § 2(b), we obtain respectively $K_{\text{alg}}(A)$, $K_{\text{f.o.}}(A)$, $K_{\text{hol}}(A)$ where $K_{\text{alg}}(A)$ and $K_{\text{hol}}(A)$ have been previously defined.

(iii) If $\Lambda \subset \Lambda'$, then there is a natural map $K_{\Lambda'}(A) \to K_\Lambda(A')$, and the *Oka principle with growth conditions*

$$K_{\text{f.o.}}(A) \cong K_{\text{hol}}(A) \tag{6.1}$$

is valid.

(iv) There is a natural map (cf. § 3(c))

$$d\log\colon K_{\text{f.o.}}(A) \to H^{\text{even}}_{DR}(A_{\text{alg}}) \tag{6.2}$$

which is an isomorphism on $K_{\text{f.o.}}(A) \otimes_{\boldsymbol{Z}} \boldsymbol{C}$.

(v) There is a natural map (cf. § 4(e))

$$\text{ch}\colon K_\Lambda(A) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \to \mathscr{C}_*(A) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}\ , \tag{6.3}$$

which is an isomorphism if $\Lambda \supseteq \{1, r\}$.

(vi) For the case of line bundles, $K_\Lambda(A)$ specializes to $\text{Pic}_\Lambda(A)$ as defined in § 5.

Remarks. The map "$d\log$" in (6.2) was discussed in § 3(c) where we gave a homomorphism

$$d\log\colon K_{\text{hol}}(A) \to H^{\text{even}}_{DR}(A_{\text{hol}})\ .$$

Recall that $K_{\text{f.o.}}(A)$ is hopefully the missing term (?) in (3.19).

The map "ch" in (6.3) is to be defined by taking the *Chern cycles* (cf. § 6(b)

[36] We shall concentrate on the case when A is affine. The general case is derived from this by "recollement".

[37] It is important to emphasize that $K_\Lambda(A)$ depends functorially on the algebraic, but generally not on the *complex*, structure on A. Thus, e.g., the biholomorphic automorphism $(z_1, z_2) \to (z_1 + e^{z_2}, z_2)$ of $\boldsymbol{C}^2$ transforms polynomials into transcendental functions. An important special case where $K_\Lambda(A)$ depends functorially on the complex structure of A is when there is a smooth completion $\bar{A}$ of A such that, for large μ,

$$\dim \{H^0(\bar{A}, \mathscr{O}(K^{\mu}_{\bar{A}}))\} \geq c\mu^n \qquad (c > 0)\ ,$$

where $K_{\bar{A}} \to \bar{A}$ is the *canonical bundle* of $\bar{A}$. This condition is independent of the smooth completion $\bar{A}$, and when it happens we will say that A is of *general type* (cf. appendix in [9]).

below) of holomorphic bundles of order Λ over A, and then taking the Chern character of these Chern cycles so as to have a ring homomorphism [3].

(a) *Tentative definition of* $K_\Lambda(A)$

Let Grass (m, N) be the Grassmann manifold of $(N - m)$-planes through the origin in C^N. Over Grass (m, N) we have the *universal quotient bundle* $Q \to$ Grass (m, N) whose fibre Q_π over a plane $\pi \in$ Grass (m, N) is just the quotient space C^N/π. There is an obvious *evaluation map*

(6.4) $$e \colon C^N \to H^0(\text{Grass}(m, N), \mathcal{O}(Q)) ,$$

which is an isomorphism of vector spaces.

(6.5) Definition. A holomorphic vector bundle $E \to A$ has order Λ if there is a holomorphic mapping $f \colon A \to$ Grass (m, N) such that: (i) f has order Λ, and (ii) $f^{-1}(Q) \cong E$.

Remarks. If E_1 and E_2 are of order Λ, then so are $E_1 \oplus E_2$ and $E_1 \otimes E_2$. However, it is by no means clear that the dual bundle E_1^* also has order Λ, and likewise for Hom (E_1, E_2). Referring to the three examples of Λ-rings in § 2(b), the first and third examples specialize respectively to the usual definitions of an algebraic vector bundle and holomorphic vector bundle.

Now we consider a smooth completion $\bar{A}$ of A and denote by $i \colon A \to \bar{A}$ the inclusion. As in § 2 we may define the subsheaf on $\bar{A}$

$$\mathcal{O}_\Lambda(A) \subset i_* \mathcal{O}(A)$$

of holomorphic functions of order Λ on A. Let $E \to$ A have order Λ. Referring to (6.4) there is a canonical mapping

(6.6) $$e \colon \mathcal{O}_\Lambda(A)^N \to i_* \mathcal{O}(E) .$$

(6.7) Definition. The sheaf $\mathcal{O}_\Lambda(E)$ of holomorphic sections of $E \to A$ of order Λ is the image of e in (6.6).

Remarks. $\mathcal{O}_\Lambda(E)$ is a sheaf of $\mathcal{O}_\Lambda$-modules. In case $E \to A$ is a line bundle, it may be shown that the above definition of $\mathcal{O}_\Lambda(E)$ coincides with the previous one given by using transition functions in § 5(b) (here Λ is restricted to the examples in § 2(b)).

(6.8) Definition. If E_1, E_2 have order Λ, then we define the subsheaf

$$\text{Hom}_\Lambda (E_1, E_2) \subset i_* \mathcal{O} (\text{Hom} (E_1, E_2))$$

by

$$\text{Hom}_\Lambda (E_1, E_2) = \text{Hom}_{\mathcal{O}_\Lambda} (\mathcal{O}_\Lambda(E_1), \mathcal{O}_\Lambda(E_2))$$

(cf. [11, p. 125]).

Using this definition, we may define an isomorphism of order Λ between E_1 and E_2, as well as Λ-exact sequences

$$(6.9) \qquad 0 \longrightarrow E' \xrightarrow{j} E \xrightarrow{p} E' \longrightarrow 0$$

involving holomorphic bundles of order Λ (both j and p should have order Λ).

(6.10) Definition. The ring $K_\Lambda(A)$ is defined to be the ring generated by Λ-isomorphism classes of holomorphic bundles of order Λ, the algebraic operations being $\oplus$ and $\otimes$, and with the relation $E = E' + E''$ whenever we have a Λ-exact sequence (6.9).

We shall be giving several comments on this definition of $K_\Lambda(A)$. For the moment, we should like to observe three simple facts. The first is that $K_\Lambda(A)$ is functorially associated to the algebraic structure on A (cf. footnote 37). The second is that, referring to the first and third examples of λ-rings in § 2(b), we obtain respectively $K_{\text{alg}}(A)$ and $K_{\text{hol}}(A)$ as defined previously. Thus properties (i) and (ii) stated at the beginning of § 6 are satisfied. The final comment is concerning the standard *Plücker embedding* [14]

$$p\colon \text{Grass}\,(m, N) \to P_{\binom{N}{m}-1}\,.$$

A holomorphic mapping $f\colon A \to \text{Grass}\,(m, N)$ induces $g = p \circ f\colon A \to P_{\binom{N}{m}-1}$ such that

$$g^{-1}(H) = \det\,[f^{-1}(Q)]$$

where $H \to P_{\binom{N}{m}-1}$ is the standard ample line bundle. It follows more or less from the definitions that f has order $\Lambda \Leftrightarrow g$ has order Λ.

(b) $K_\Lambda(A)$ *and Chern cycles*

Let A be a smooth affine variety, $f\colon A \to \text{Grass}\,(m, N)$ a holomorphic mapping, and $E = f^{-1}(Q)$ the inverse image of the universal quotient bundle over Grass (m, N). Then there are N distinguished holomorphic sections $\sigma_1, \cdots, \sigma_N$ of $Q \to A$ corresponding to (6.4).[38]

(6.11) Definition. The *q-th Chern cycle* $V_q(E)$ is the analytic subvariety of A where $\sigma_1 \wedge \cdots \wedge \sigma_{m-q+1} = 0$.

Remarks. Geometrically, $V_q(E)$ is the intersection of the image $f(A)$ with a *Schubert cycle* Σ_q in Grass (m, N) [14]. $V_q(E)$ is an analytic subvariety of pure codimension q on A (cf. footnote 38 and [8, p. 225]). The extreme cases are: (i) $V_m(E)$ is the set of zeroes of the section σ_1 of $Q \to A$; and (ii) $V_1(E)$ is the divisor of the section $\sigma_1 \wedge \cdots \wedge \sigma_m$ of $\det(Q) = \Lambda^m Q$.

[38] It will make this discussion easier if we assumed that $\sigma_1, \cdots, \sigma_N$ generate an *ample* space of sections of $Q \to A$ (cf. [8, p. 185]). We may always twist Q by an ample, algebraic line bundle to arrive at this situation.

Problem B. If Λ is a λ-ring, and f has order Λ, then does $V_q(E) \in \mathscr{C}_q(A_\Lambda)$?[39]

Remarks. This is the analogue of the basic implication Ⓐ ⇒ Ⓑ in Theorem of § 5(a). We shall reduce Problem B to either of two semi-local statements in several complex variables.

First observe that, by the argument using Jensen's theorem given in § 5(b) (cf. footnote 31), Problem B is true for $q = 1$ (case of divisors). Since each Schubert cycle Σ_q is contained in an intersection of hypersurfaces on Grass (m, N), Problem B follows from Problem A in § 4(e). Problem A follows in turn from either of the following two questions:

(6.12) Question. Let $\eta_1, \cdots, \eta_q$ be holomorphic functions on a punctured polycylinder P^* such that

$$M(\eta_j, r) = 0(\lambda(r)) \qquad (\lambda \in \Lambda) .$$

Let $V = \{\eta_1 = \cdots = \eta_q = 0\}$ be an analytic subvariety, assumed to have pure codimension q in P^*. Then do we have

$$N(V, r) = 0(\lambda'(r)) \qquad (\lambda' \in \Lambda)?$$

(6.13) Question. Let $V \subset \boldsymbol{C}^n$ be an analytic set of pure codimension q and assume that

$$\int_{V[r]} \omega^{n-q} = 0(\lambda(r))$$

for some $\lambda \in \Lambda$ (cf. Question (4.14)). Suppose that $\boldsymbol{C}^l \subset \boldsymbol{C}^n$ is a linear subspace such that the intersection $W = V \cdot \boldsymbol{C}^l$ has pure codimension q in $\boldsymbol{C}^l$. Then do we have

$$\int_{V[r]} \omega^{l-q} = 0(\lambda'(r))$$

for some $\lambda' \in \Lambda$?

Remark. These questions are essentially the same as "Bezout's problem" discussed in Appendix 2 to [9].

In concluding this section we observe that the resolution of Problem B is necessary in order that the map "ch" in (6.3) be defined. At the risk of being overly categorical, we might say that: *Jensen's theorem for algebraic varieties would be the assertion that the map*

$$\mathrm{ch}\colon K_\Lambda(A) \to \mathscr{C}_*(A_\Lambda)$$

is well-defined.

[39] Cf. § 4(e) for the definitions. What we are asking is whether: "f order $\Lambda \Rightarrow V_q(E)$ order Λ"? We shall see below that problems A (§ 4(e)) and B are special cases of the same problem.

(c) *Coherent sheaves with Λ-growth conditions*

In the preceding section we have discussed a problem whose solution would say that the analytic subvarieties arising from a holomorphic bundle of order Λ would themselves have order Λ. We shall now discuss the converse question.

(6.14) Definition. Let S on $\bar{A}$ be a sheaf of $\mathcal{O}_\Lambda(A)$-modules. Then S is said to be a *coherent Λ-sheaf* if, locally on $\bar{A}$, we have a resolution by free $\mathcal{O}_\Lambda(A)$-modules

$$(6.15) \qquad 0 \longrightarrow \mathcal{O}_\Lambda(A)^{(m_1)} \xrightarrow{\tau_1} \mathcal{O}_\Lambda(A)^{(m_2)} \xrightarrow{\tau_2} \cdots \longrightarrow \mathcal{O}_\Lambda(A)^{(m_l)} \longrightarrow S \longrightarrow 0$$

where the maps τ_α have order Λ.

Problem C. Let $V \subset A$ be an analytic subvariety of order Λ. Then is there a coherent Λ-sheaf $\mathcal{I}_\Lambda \subset \mathcal{O}_\Lambda(A)$ such that

$$\text{Support}\,(\mathcal{O}_\Lambda(A)/\mathcal{I}_\Lambda) = V\ ?$$

Problem D. Let S be a coherent Λ-sheaf on A. Then is there a global resolution

$$(6.16) \qquad 0 \to \mathcal{O}_\Lambda(E_1) \xrightarrow{\tau_1} \mathcal{O}_\Lambda(E_2) \xrightarrow{\tau_2} \cdots \longrightarrow \mathcal{O}_\Lambda(E_l) \longrightarrow S \longrightarrow 0$$

where the $E_\alpha \to A$ are holomorphic vector bundles of order Λ?

Remarks. In problem C we are not requiring that the sections of $\mathcal{I}_\Lambda$ should generate the ideal of V at every point, as this is probably impossible to achieve (cf. footnote 21).

Next, we observe that the proof of Ⓑ $\Rightarrow$ Ⓒ in Theorem I also yields a proof of Problem C in case codim $(V) = 1$. Futhermore, the proof of the implication Ⓒ $\Rightarrow$ Ⓐ in Theorem I leads to a proof of the last step in Problem D, also in the case of divisors.

As somewhat scant evidence that these problems might be possible in higher codimension, we remark that the recent paper of Pan [20] gives us a sheaf mapping

$$\mathcal{O}_\Lambda(A)^{(n+1)} \xrightarrow{\tau} \mathcal{O}_\Lambda(A)$$

such that

$$\text{supp}\,\{\mathcal{O}_\Lambda(A)/\tau[\mathcal{O}_\Lambda(A)^{(n+1)}]\} = V$$

in the case where $\dim_C V = 0$ (i.e. V is a discrete set of points with order Λ).

The resolution of Problems B, C, and D are essentially what is necessary in order that the mapping "ch" in (6.3) be defined and be an isomorphism.

(d) *Transition functions for bundles of order Λ*

We have given a definition of what it means for a holomorphic vector bundle $E \to A$ to have order Λ.[40] For line bundles, there are three equivalent definitions which, in the case of vector bundles, may be informally stated as follows:

Ⓐ $\leftrightarrow$ {E is induced from a holomorphic mapping $f: A \to \text{Grass}(m, N)$ which has order Λ};

Ⓑ $\leftrightarrow$ {the Chern cycles of E are analytic subvarieties of order Λ in A};

Ⓒ $\leftrightarrow$ {relative to finite covering of A by punctured polycylinders, E has transition functions of order Λ}.

Actually, we have not formally defined Ⓒ, so let us give the

(6.17) Definition. Let P^* be a punctured polycylinder. Then $GL(m, \mathcal{O}_\Lambda(P^*))$ is the group of holomorphic mappings $g: P^* \to GL(m, C)$ which have order Λ in the sense that

$$g \cdot \mathcal{O}_\Lambda(P^*)^m \subset \mathcal{O}_\Lambda(P^*)^m \ .$$

Remarks. It follows that $GL(m, \mathcal{O}_\Lambda(P^*))$ is just the group of matrices whose entries are in the ring $\mathcal{O}_\Lambda(P^*)$. For the λ-rings under consideration, we shall denote $GL(m, \mathcal{O}_\Lambda(P^*))$ respectively by

$$GL(m, \mathcal{O}_{\text{alg}}(P^*)) \ , \quad GL(m, \mathcal{O}_{\text{f.o.}}(P^*)) \ , \quad GL(m, \mathcal{O}_{\text{hol}}(P^*)) \ .$$

Under Ⓒ above, we mean that the transition functions at infinity of $E \to A$ are in $GL(m, \mathcal{O}_\Lambda(E))$.[41]

Problem. Are the above three definitions of a holomorphic vector bundle of order Λ all equivalent?

Remarks. The implication Ⓐ $\Rightarrow$ Ⓑ would follow from Problem B in § 6(b). The implication Ⓑ $\Rightarrow$ Ⓒ would most likely be a consequence of Problem C, also in § 6(b).[42] Finally, the implication Ⓒ $\Rightarrow$ Ⓐ can probably be proved by the same methods used to demonstrate this result in the case of line bundles (cf. § 5(b)). Consequently, this problem does not appear to go much beyond those stated previously, and so we have not given it a separate letter.

Our final problem concerns the Lie algebra of the group $GL(m, \mathcal{O}_\Lambda(P^*))$. Given a holomorphic mapping $f: P^* \to GL(m, C)$, we let

$$d \log f = df \cdot f^{-1} \ .$$

[40] In this section we shall assume that Λ is one of the three λ-rings in § 2(b).

[41] Referring to definition C in § 5(a), we are here restricted to the case $A = \bar{A} - D$ where D is smooth (cf. footnote 29).

[42] In discussing the implication Ⓑ $\Rightarrow$ Ⓒ, we are tacitly assuming that E is determined by its Chern cycles. To be precise then, this implication should be interpreted as saying that the holomorphic vector bundle $\underbrace{E \oplus \cdots \oplus E}_{\mu} \oplus I$ has transition functions of order Λ.

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